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REDUCTION OF AN ELASTIC SYSTEM TO A PRESCRIBED STATE BY USE OF A BOUNDARY FORCE*

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A solution is constructed for the problem of optimal control of the motion of a distributed elastic system, using a lumped boundary force. The system's state is described by a constant-coefficient hyperbolic equation. A general case of arbitrary initial and final distributions is examined. Questions of control by lumped and distributed forces are discussed.

1. Introductory remarks and statement of the fundamental control problem.

We examine the optimal control problem for the state of a homogeneous elastic system on a fixed time interval /1,2/. We assume that the initial and the final distributions of the elastic deviations are specified and arbitrary. For definiteness we can consider a homogeneous elastic spring (beam) or an elastic shaft as the mechanical analog of such a system. As the controlling time function we take the magnitude of a force (a force or a moment of forces) on the system's boundary. The state of such a system is described by a hyperbolic partial differential equation and can be investigated under prescribed forces within the framework of the theory of the equations of mathematical physics /3/. The integral of the square of the control function serves as the functional characterizing the control's performance. Such a statement of the control problem enables us to answer questions on the principal capabilities of the control in the class of bounded piecewise-continuous time functions. We note that under certain conditions the statement given can be ill-posed /4/. To construct the piece-wise-continuous control we use a modified method of moments /5,6/ analogous to that used in /1/. As a result we find the optimal control problem's solution in explicit finite form.

The investigation of control problems for systems with elastic elements by means of boundary, distributed, lumped (impulsive) and moving controlling forces is urgent at the present time for solving a number of applied problems in mechanics and engineering. Theoretically the solution of control problems for distributed-parameter oscillating systems is beset by a number of specific singularities /1,7/ in comparison with the corresponding finite-dimensional analogs. In various settings such problems were investigated in /1,2,7—13/ and elsewhere.

We state the fundamental control problem. We assume that the system to be controlled (elastic beam or shaft) is described by the equations /3/

$$\rho\varphi^{\cdot\cdot} = c\varphi'', \quad \varphi = \varphi(t, x), \quad x \in (0, l), \quad c\varphi'(t, 0) = -M(t), \quad \varphi'(t, l) = 0, \quad t \in [t_0, T] \tag{1.1}$$

Here φ is the elastic deviation of the cross-section (linear or angular) with coordinate x at instant t; the dots indicate differentiation with respect to t and the primes, with respect to x. The problem's constant parameters are: ρ is the linear inertia characteristic (distribution density of the mass or of the moment of inertia), c is the material's rigidity characteristic (the Young's modulus or the torsional rigidity), l is the length, $\rho, c, l > 0$, t_0, T are prescribed instants, $T > t_0$. The unknown function $M(t)$ is the control (a force or a moment of forces, lumped, for definiteness, at the left end $x = 0$).

We pose the optimal control problem for the distributed system (1.1) being controlled. By a choice of the admissible control function $M(t)$ we take the system from an arbitrary initial state

$$\varphi(t_0, x) = f^\circ(x), \quad \varphi^\cdot(t_0, x) = g^\circ(x) \tag{1.2}$$

to a prescribed final state

$$\varphi(T, x) = f^T(x), \quad \varphi^\cdot(T, x) = g^T(x) \tag{1.3}$$

in such a way that the functional

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$$J[M]=\int_{t_0}^{T} M^2(t)\,dt \to \min_M, \quad |M|<\infty \tag{1.4}$$

is minimized /1/. Here $f^{0,T}(x)$, $g^{0,T}(x)$ are sufficiently smooth functions of $x, x \in [0, l]$, continuous to be precise, and $f^{0,T}$ is piecewise-continuously differentiable. We shall establish below that the optimal control problem (1.1)—(1.4) has a solution when certain additional assumptions are fulfilled. We pass to dimensionless quantities by means of the linear transformations (see /11/)

$$\begin{gathered} t_* = \nu t, \quad \nu^2 = c / \rho l^2, \quad x_* = x / l, \quad x_* \in [0, 1] \\ \varphi_* = \varphi / L, \quad f_*^{0,T}(x_*) = f^{0,T}(x_* l) / L \\ g^{0,T}(x_*) = g^{0,T}(x_* l) / \nu L, \quad M_* = Ml / cL, \quad J_* = J\nu l^2 / c^2 L^2 \end{gathered} \tag{1.5}$$

Here L is a typical magnitude of the dimension of variable φ; in particular, we can set $L = l$, for elastic linear displacements and $L = 1$ for torsional ones. Henceforth we omit the subscript $*$ to shorten the notation and we obtain the relations of optimal control problem (1.1) — (1.4), in which $\rho = c = l = 1$.

Because the problem is autonomous (stationary), the time dependence in the control enters as a difference $t - t_0$ and $T - t_0$. Therefore, it can be examined for $t_0 = 0$, with a subsequent substitution $t \to t - t_0$, $T \to T - t_0$. The final conditions (1.3) permit us, in particular, to treat the elastic system's displacement problem as a whole at a specified distance or at a fixed point ξ with a damping of the oscillations

$$\varphi(T, x) = \xi, \quad \varphi^{\cdot}(T, x) = 0 \tag{1.6}$$

The problem of delineating some osillation mode can be posed analogously (see Sect.3). Other problem statements having a definite mechanical sense are possible. For instance, the reduction of the system to a state of translational displacement with velocity η (without elastic oscillations):

$$\varphi(T, x) - \langle\varphi(T, x)\rangle = 0, \quad \varphi^{\cdot}(T, x) = \eta \tag{1.7}$$

Here the angle brackets denote the average with respect to x, $x \in [0, 1]$. Final conditions that are combinations of (1.6) and (1.7) are of specific interest. In the general case, the final conditions imposed on $\varphi, \varphi^{\cdot}$ can be a system (finite or countable) of functionals on $\varphi(T, x)$, $\varphi^{\cdot}(T, x)$.

2. Solution of the fundamental problem. We apply the Fourier method /1,3,11/, i.e., we construct the function $\varphi(t, x)$ in the form

$$\varphi(t, x) = \sum_{n=0}^{\infty} T_n(t) X_n(x), \quad X_n(x) = \cos \pi n x, \quad n = 0, 1, \ldots \tag{2.1}$$

Here $\{X_n\}$ is the orthogonal system of eigenfunctions of the homogeneous boundary-value problem (1.1) and $T_n(t)$ are unknown functions of t, yet to be determined. They form the solution of the infinite-dimensional optimal control problem $(n = 0, 1, 2, \ldots)$:

$$\begin{gathered} T_0^{\cdot\cdot} = M(t), \quad T_n^{\cdot\cdot} + \pi^2 n^2 T_n = 2M(t), \quad t \in [0, T] \\ T_n|_{t=0, T} = f_n^{0,T}, \quad T_n^{\cdot}|_{t=0, T} = g_n^{0,T} \\ J[M] = \int_0^T M^2(t)\,dt \to \min_M, \quad |M| < \infty \end{gathered} \tag{2.2}$$

The constants $f_n^{0,T}$, $g_n^{0,T}$ are the Fourier coefficients of the functions $f^{0,T}(x)$, $g^{0,T}(x)$ from (1.2), (1.3) (or (1.6) or (1.7)) with respect to the system $\{X_n\}$. Applying the procedure in /2/, from the form of the solution of the adjoint problem we find that the optimal control $M(t)$ is

$$M(t) = At + v(t), \quad v(t+2) = v(t), \quad t \in [0, T] \tag{2.3}$$

Here A is an unknown constant and $v(t)$ is an unknown 2-periodic function of t, representable by a Fourier series. To determine the unknown quantities we make use of the representations of functions $T_n(t)$ and of their derivatives, in accordance with (2.2)

$$T_0(t)=\int_0^t (t-\tau)M(\tau)\,d\tau+f_0^\circ+g_0^\circ t,\quad T_0^{\boldsymbol{\cdot}}=\frac{dT_0}{dt} \tag{2.4}$$

$$T_n(t)=\frac{2}{\pi n}\int_0^t M(\tau)\sin\pi n(t-\tau)\,d\tau+f_n^\circ\cos\pi nt+\frac{g_n^\circ}{\pi n}\sin\pi nt,\qquad T_n^{\boldsymbol{\cdot}}=\frac{dT_n}{dt},\quad n=1,2,\ldots$$

Substituting functions (2.4) into the final conditions (2.2), for the determination of the unknown function we have the moments problem

$$\int_0^T v(\tau)\,d\tau=g_0^T-g_0^\circ-\frac{1}{2}AT^2 \tag{2.5}$$

$$\int_0^T v(\tau)\cos\pi n\tau\,d\tau=\frac{\pi n}{2}f_n^T\sin\pi nT+\frac{1}{2}g_n^T\cos\pi nT-\frac{1}{2}g_n^\circ-A\int_0^T\tau\cos\pi n\tau\,d\tau$$

$$\int_0^T v(\tau)\sin\pi n\tau\,d\tau=-\frac{\pi n}{2}f_n^T\cos\pi nT+\frac{1}{2}g_n^T\sin\pi nT+\frac{\pi n}{2}f_n^\circ-A\int_0^T\tau\sin\pi n\tau\,d\tau$$

Since $v(t)$ is a periodic function, setting /1/ $T=2N+\theta$, $N=0,1,2,\ldots$, $0\leqslant\theta<2$, we obtain for the new unknown function

$$V_T(t)=\begin{cases}(N+1)v(t), & t\in[0,\theta]\\ Nv(t), & t\in(\theta,2)\end{cases} \tag{2.6}$$

an explicit expression in the form of the Fourier series

$$V_T(t)=G_0+\sum_{n=1}^{\infty}(G_n\cos\pi nt+F_n\sin\pi nt) \tag{2.7}$$

$$G_0=g_0^T-g_0^\circ-{}^1/_2AT^2$$

$$G_n=\frac{\pi n}{2}f_n^T\sin\pi n\,T+\frac{1}{2}g_n^T\cos\pi nT-\frac{1}{2}g_n^\circ-A\int_0^T\tau\cos\pi n\tau\,d\tau$$

$$F_n=-\frac{\pi n}{2}f_n^T\cos\pi nT+\frac{1}{2}g_n^T\sin\pi nT+\frac{\pi n}{2}f_n^\circ-A\int_0^T\tau\sin\pi n\tau\,d\tau$$

On the basis of (1.2), (1.3), (2.2) the periodic function $V_T(t)$ is formally determined, according to (2.7), in explicit form for all t:

$$V_T(t)=\Phi(T,t)-A\Psi(T,t) \tag{2.8}$$

$$\Phi(T,t)=\frac{1}{2}G^T\left(T-t-2\left[\frac{T-t}{2}\right]\right)+F^{T'}\left(T-t-2\left[\frac{T-t}{2}\right]\right)\Big\}-\frac{1}{2}\Big\{G^\circ\left(t-2\left[\frac{t}{2}\right]\right)+F^{\circ'}\left(t-2\left[\frac{t}{2}\right]\right)\Big\}$$

$$\Psi(T,t)=t+2\left[\frac{T-t}{2}\right]\frac{t+1}{2}+2\left[\frac{t}{2}\right]\frac{t}{2}+\frac{1}{4}\left(2\left[\frac{T-t}{2}\right]\right)^2-\frac{1}{4}\left(2\left[\frac{t}{2}\right]\right)^2$$

Here the square brackets denote the integer part of the number. We note that the functions Φ, Ψ are 2-periodic, while the functions $G^{0,T}$, $F^{0,T}$ are determined, by use of functions $g^{0,T}$, $f^{0,T}$, by the relations /1/

$$G^{0,T}(t)=\begin{cases}g^{0,T}(t), & 0\leqslant t\leqslant 1\\ g^{0,T}(2-t), & 1<t\leqslant 2\end{cases} \tag{2.9}$$

$$F^{0,T}(t)=\begin{cases}f_x^{0,T'}(t), & 0\leqslant t\leqslant 1\\ -f_x^{0,T'}(2-t), & 1<t\leqslant 2\end{cases}$$

We can now find the function $v(t)$ directly from (2.6) with $N=1,2,\ldots$ in the form

$$v(t)=\begin{cases}(N+1)^{-1}V_T(t), & t\in[0,\theta]\\ N^{-1}V_T(t), & t\in(\theta,2)\end{cases} \tag{2.10}$$

However, if $N=0$, i.e., $T=\theta<2$, then function $v(t)$ may not exist for every one of the right-hand sides of relations (2.5).

Let us first consider the case $N\geqslant 1$. We find the unknown constant A from the final condition (2.2) for $T_0(T)$, using expression (2.4) into which the function $M(t)$ from (2.3) is substituted and taking (2.6), (2.8) into account. We obtain

$$M(t)=A\Psi^*(T,t)+\Phi^*(T,t),\quad t\in[0,T] \tag{2.11}$$

Here the functions Ψ^* and Φ^* for $t\in[2m,2m+\theta]$ and $t\in(2m+\theta,2(m+1))$, respectively, equal ($m=0,1,\ldots,N$):

$$\Psi^*=\begin{cases}t-(N+1)^{-1}\Psi\\ t-N^{-1}\Psi\end{cases},\qquad \Phi^*=\begin{cases}(N+1)^{-1}\Phi\\ N^{-1}\Phi\end{cases} \tag{2.12}$$

We substitute the function $M(t)$ from (2.11) into the unused final condition of (2.2): $T_0(T)=f_0^T$. The constant A is determined uniquely under the condition

$$\Delta(T)=\int_0^T (T-\tau)\Psi^*(T,\tau)\,d\tau\neq 0 \tag{2.13}$$

Substituting this value of A into (2.11), we obtain the control, optimal in the sense of criterion (1.4),

$$M^*(t)=\left[f_0^T-f_0^\circ-g_0^\circ T-\int_0^T(T-\tau)\Phi^*(T,\tau)d\tau\right]\frac{\Psi^*(T,t)}{\Delta(T)}+\Phi^*(T,t) \tag{2.14}$$

leading the elastic system (1.1) in time T from the arbitrary initial state (1.2) into the prescribed final state (1.3). The optimal controlled motion $\varphi^*(t,x)$ equals

$$\varphi^*(t,x)=\frac{1}{2}\int_0^t\{[x+(t-\tau)]-[x-(t-\tau)]\}M^*(\tau)\,d\tau+ \tag{2.15}$$

$$\frac{1}{2}\int_0^t\{g^\circ(x+\tau-[x+\tau])+g^\circ(x-\tau-[x-\tau])\}\,d\tau+\frac{1}{2}\{f^\circ(x+t-[x+t])+f^\circ(x-t-[x-t])\}$$

We note that condition (2.13) $\Delta(T)\neq 0$ is fulfilled for $T>2$ since $\Psi^*(T,t)>0$ for $t>0$. If, however, $T=2$, i.e., $N=1$, $\theta=0$, then $\Psi^*\equiv 0$ for $t\in(0,T)$ and $\Delta(T)=0$. In this case the right-hand sides of (2.5) are the Fourier coefficients of function $v(t)$, while the solution of the control problem exists when the expression within square brackets in (2.14) vanishes. Control $M(t)$ from (2.11) and functional J from (1.4) or (2.2) are independent of A.

Let us consider the moments problem (2.5) in the case $T<2$. On the complete interval $0\leqslant t\leqslant 2$ we introduce a function $w(t)$ equal to $v(t)$ when $t\in[0,\theta]$ and $w(t)\equiv 0$ when $t\in(\theta,2]$. Then, according to (2.8), we obtain $w(t)=\Phi(T,t)-A\Psi(T,t)$, where $\Psi\equiv t$; therefore, $\Psi^*(T,t)\equiv 0$ in (2.11)–(2.13), but the constant A can be arbitrary. In this connection the functions $f^{\circ,T}$ $g^{\circ,T}$ must be such that $w(t)\equiv 0$ for some A when $t\in[\theta,2]$. As mentioned in Sect.1, the control $M^*(t)$ from (2.14) has been constructed for an arbitrary initial distribution (1.2) at an arbitrary initial instant t_0. For this it is necessary to make in (2.14) the change: $t\to(t-t_0)$, $T\to(T-t_0)$. As a result the control M^* can be represented as a function of these arguments and as a linear operator on $\varphi(t_0,x)=f^\circ(x)$, $\varphi^\cdot(t_0,x)=g^\circ(x)$ (as well as on f^T, g^T; this dependency is not shown for brevity):

$$M^*=M_0(t-t_0,T-t_0,[f^\circ(x)],[g^\circ(x)]) \tag{2.16}$$

Having replaced in (2.16) the initial quantities by the current ones: $t_0\to t$, $f^\circ(x)\to\varphi(t,x)$, $g^\circ(x)\to\varphi^\cdot(t,x)$, we obtain the control, optimal in the sense of (1.4), in the form of a synthesis. We should bear in mind here that representation (2.16) is valid for $T-t>2$. Beginning with the instant $t<T-2$ we should apply the programmed control constructed above in accordance with (2.14). However, if $T-t_0<2$, then the control is of form (2.11).

3. Concrete statements of the control problem. **1°.** For the problem of damping of the elastic oscillations due to the initial deviations (1.2), the final conditions (2.2) are ($T_0(T)$, $T_0^\cdot(T)$ are arbitrary)

$$T_n(T)=0,\quad T_n^\cdot(T)=0,\quad n=1,2,\ldots \tag{3.1}$$

If the elastic body must be at rest in an arbitrary position, then conditions (3.1) must be augmented by the equality $T_0^\cdot(T)=0$, i.e., $g^T(x)\equiv 0$; in this case $A^*=0$. If the position is fixed for $t=T$, then $T_0(T)=\xi$, where ξ is a prescribed quantity, i.e., $f^T(x)=\xi$, (see (1.6)). We note that when $M\equiv 0$ $(t>T)$ the system (1.1) admits of a solution of the form indicated above.

2°. In the problem of picking out oscillation modes (see Sect.1), for example, some k-th mode, the final conditions (2.2) have the form

$$T_n(T)=0,\quad T_n^\cdot(T)=0,\quad n=0,1,\ldots,k-1,k+1,\ldots \tag{3.2}$$
$$T_k(T)=f_k^T,\quad T_k^\cdot(T)=g_k^T$$

Obviously, the functions f^T and g^T equal

$$f^T(x)=f_k^T\cos\pi kx,\quad g^T(x)=g_k^T\cos\pi kx$$

If, further, we set the control $M(t)\equiv 0$ when $t>T$, then the elastic system will accomplish oscillations of only the k-th mode.

3°. Let us consider the case $\theta=0$, i.e. the time interval $T=2N$, $N=1,2,\ldots$, is a multiple of the period of the natural oscillations. For $N=1$ the problem has been studied in Sect.2. For $N=2,3,\ldots$ the construction of control $M(t)$ on the basis of (2.11) is essentially simplified. In accord with (2.6)—(2.10) the function $v(t)$ is specified by one expression: $v(t)=V_T(t)/N$, $t\in[0,2N]$. The control $M(t)$ too is computed uniquely with the aid of (2.15). In particular, in the problem of leading the elastic system to a state of translational displacement as a whole with prescribed velocity η (see (1.1), (1.2), (1.7)) the optimal control equals

$$M^*(t)=\frac{\eta}{2N}-\frac{1}{2N}\left\{G^\circ\left(t-2\left[\frac{t}{2}\right]\right)+F^{\circ\cdot}\left(t-2\left[\frac{t}{2}\right]\right)\right\} \tag{3.3}$$

The optimal motion is described by formula (2.16). The solution of the problem of leading the elastic system to a prescribed position $\varphi(T,x)=\xi$ (see 1.1), (1.2), (1.6)) with a damping of the elastic oscillations is constructed analogously.

4. Generalization of the control problem. The investigation of the control problem for a system more general than (1.1) is of systematic interest. The elastic system can be subject to additional controls and to external forces both distributed as well as lumped (boundary)

$$\rho\varphi^{\cdot\cdot}=c\varphi''+w(t,x)+W,\quad \varphi=\varphi(t,x),\quad x\in(0,l) \tag{4.1}$$
$$c\varphi'(t,0)=-m_0(t)-M_0,\quad c\varphi'(t,l)=m_l(t)+M_l,\quad t\in[t_0,T]$$

Here w is a prescribed distributed external force, m_0, m_l are prescribed forces lumped at the left and right ends, W is a distributed control, a function of t and x, M_0, M_l are lumped (boundary) controls, functions of t (see Sect.2). The initial and final conditions for the variable $\varphi=\varphi(t,x)$ again are of form (1.2), (1.3), or more general (see Sect.1). As the functional to be minimized we can take a weighted sum of quantities of form (1.4)

$$J[W,M_0,M_l]=c_W^2\int_{t_0}^{T}dt\int_0^l W^2\,dx+\int_{t_0}^{T}(c_0^2M_0^2+c_l^2M_l^2)\,dt\to\min_{W,M_0,M_l} \tag{4.2}$$

Here c_W^2, c_0^2, c_l^2 are constant "weight" coefficients greater than zero. An unrestricted increase of some of them makes the corresponding controls tend to zero. and their contributions to functional (4.2) tend to zero as well. For instance, as $c_{0,l}^2\to\infty$ we obtain the optimal control problem for system (4.1), (1.2), (1.3) with a functional (4.2) in which $M_0=M_l\equiv 0$, i.e., the control is effected solely by the function $W(t,x)$. Analogously to Sect.2 we can establish that the solution exists and is unique for any sufficiently smooth functions w, m_0, m_l, $f^{0,T}$, $g^{0,T}$ and $T>t_0$. Its explicit construction both as a program and as a synthesis presents no difficulty. Indeed, setting $s=t-t_0$, where $s\in[0,S]$, $S=T-t_0$, by the Fourier method, having solved the adjoint system /2/, we obtain the expression

$$W(t,x) = \frac{1}{2}(A_0 s + B_0) + \sum_{n=1}^{\infty} (A_n \sin \pi n s + B_n \cos \pi n s) \cos \pi n x \tag{4.3}$$

for the required control $W(t, x)$. The constants A_n, B_n $(n = 0, 1, \ldots)$ are determined from final conditions of type (2.2):

$$\begin{aligned} A_n &= \frac{1}{\Delta_n(S)} \left[\frac{a_n}{2} \left(S + \frac{\sin 2\pi n S}{2\pi n} \right) - b_n \frac{\sin^2 \pi n S}{2\pi n} \right] \\ B_n &= \frac{1}{\Delta_n(S)} \left[\frac{b_n}{2} \left(S - \frac{\sin 2\pi n S}{2\pi n} \right) - a_n \frac{\sin^2 \pi n S}{2\pi n} \right] \\ \Delta_n(S) &= (S^2/4)[1 - (\pi n S)^{-2} \sin^2 \pi n S] \end{aligned} \tag{4.4}$$

The expressions for A_0, B_0 are obtained from (4.4) as $n \to 0$. The coefficients a_n, b_n depend on the parameters t_0, S and are functionals on the initial and final distributions, as well as on the prescribed boundary and distributed forces

$$a_n = -\int_0^S [w_n(\sigma + t_0) + 2m_0(\sigma + t_0) + 2m_l(\sigma + t_0)] \sin \pi n \sigma \, d\sigma + \pi n f_n^{\circ} - \pi n f_n^T \cos \pi n S + g_n^T \sin \pi n S$$

$$b_n = -\int_0^S [w_n(\sigma + t_0) + 2m_0(\sigma + t_0) + 2m_l(\sigma + t_0)] \cos \pi n \sigma \, d\sigma - g_n^{\circ} + \pi n f_n^T \sin \pi n S + g_n^T \cos \pi n S$$

$$w(t, x) = \sum_{n=0}^{\infty} w_n(t) \cos \pi n x, \quad t = s + t_0$$

A practical realization of such distributed controls is very difficult. In practice, obviously, lumped (impulsive) forces can be realized on some, possibly dense enough, set of points of the distributed elastic system. In such situations there arise the problems which were discussed in /12/. The case of a moving control of distributed-parameter systems was examined in /7/.

Of significant applied interest is also the study of the optimal control problem in the case when the density ρ and the rigidity c are variable. For example, the system being controlled can contain absolutely rigid bodies (flywheels /11/) or irregularly distributed masses. For such problems we can use an approach analogous to the one presented, which leads, however, to a more complex moments problem (see /1,7/). However, if the system is close to the homogeneous form (4.1), then for an approximate solution of the boundary-value problem it is possible to apply the perturbation method /14/ on the basis of which an approximate optimal control can be constructed.

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